

Dirac equation on a G_2 manifold

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Abstract

We find a large family of solutions to the Dirac equation on a manifold of G_2 holonomy asymptotic to a cone over $S^3 \times S^3$, including all radial solutions. The behaviour of these solutions is studied as the manifold develops a conical singularity. None of the solutions found are both localised and square integrable at the origin. This result suggests that M-theory on the conifold over $S^3 \times S^3$ does not produce a chiral theory. The approach here is complementary to previous analyses using dualities and anomaly cancellation which found chiral fermions on all known G_2 conifolds except that over $S^3 \times S^3$.

Eleven dimensional supergravity cannot give a chiral four dimensional theory upon compactification on a smooth manifold [1]. One way to resolve this discrepancy with experiment is by compactifying on a manifold with boundary [2, 3]. Recently, work has centred on obtaining chiral fermions by the alternative approach of compactifying on singular manifolds. As supergravity is no longer valid near the singular points, the arguments make essential use of M-theory. Given that M-theory remains unformulated, this means they work via dualities with string theories. Approaches include duality with intersecting D6-branes and orientifolds of type IIA string theory [4, 5, 6] and fibrewise duality with Heterotic string theory compactified on certain Calabi-Yau manifolds [7]. The duality results are corroborated by Witten's analysis of anomaly cancellation at conical singularities [8].

A recurrent theme in these arguments is compactification on manifolds of G_2 holonomy ([9] for mathematical background, [10] for a pedagogical introduction). This is unsurprising as supersymmetry plays an important rôle in duality arguments and G_2 holonomy is the condition on the internal seven dimensional manifold for compactification from eleven to four dimensions to preserve $\mathcal{N} = 1$ supersymmetry. The preservation of $\mathcal{N} = 1$ supersymmetry is phenomenologically appealing and G_2 manifolds were considered in early work on Kaluza-Klein supergravity [11, 12]. The possibility of further obtaining chiral theories with non-abelian gauge symmetry via M-theory is even more phenomenologically interesting.

Parallel to the renewed interest in the physics of G_2 holonomy compactifications, there has been a lot of recent work on constructing and classifying G_2 metrics. All the examples known are cohomogeneity-one metrics on noncompact spaces that are asymptotic to generalised cones [13, 14, 15, 16, 17, 18, 19, 20, 21, 22]. The metrics constructed are generalisations of the metrics found a decade ago by [23, 24]. They are asymptotic to cones over $SU(3)/U(1) \times U(1)$, \mathbb{CP}^3 and $S^3 \times S^3$. No explicit metrics are known on compact G_2 manifolds because these cannot have continuous isometries except for trivial S^1 factors. The appearance of chiral fermions is a phenomenon localised near the conical singularity and so considering noncompact manifolds should not be problematic.

The original argument against obtaining chiral theories from eleven dimensions [1] is circumvented by the fact that the non-abelian gauge groups under which the chiral fermions are charged do not come from isometries of the internal manifold, but from massless branes wrapping collapsed homology cycles in the internal manifold [25, 26]. It would be nice to see these new massless fermionic degrees of freedom emerge as homology cycles collapse directly within an eleven dimensional framework, without using dualities. Further, for the G_2 manifold asymptotic

although other aspects of M-theory on this space have been studied in some detail (e.g. [4, 27]). There are various reasons for this: (a) The anomaly analysis of [8] is not applicable as for this space $H^2(X; U(1)) = 0$. (b) The G_2 manifolds considered in [7] were \mathbb{R}^3 bundles over a self-dual four-fold whilst X is an \mathbb{R}^4 bundle over S^3 . (c) Whilst the cones over $SU(3)/U(1) \times U(1)$ and \mathbb{CP}^3 were interpreted in [4] as the M-theory lift of intersecting D6-branes, the analogous interpretation for the $S^3 \times S^3$ case is not clear.

The line of attack in this work will be to look for solutions to the massless Dirac equation on manifolds on G_2 holonomy and see how the behaviour of the solutions changes as the manifold develops a conical singularity. We use the Dirac equation and not the Rarita-Schwinger spin-3/2 equation because the degrees of freedom we are interested in do not originate from supergravity but from wrapped branes. It turns out that the G_2 manifold on which the Dirac equation is most tractable, allowing us to find all radial solutions and several non-radial solutions, is precisely the one asymptotic to a cone over $S^3 \times S^3$. Thus the information we gain through this approach is nicely complementary to the information already obtained through dualities. Solving the Dirac equation on other G_2 manifolds could also be interesting, but is not done here.

In §2 we write down the massless Dirac equation on the G_2 manifold and find all radial solutions by directly solving the equation. In §3 we find some non-radial solutions by relating solutions of the massless Dirac equation to solutions of the massless Klein-Gordon equation via the covariantly constant spinor on the G_2 manifold. The qualitative behaviour of the solutions does not change in the singular limit and none of the solutions have the property of being both localised at the origin and square integrable at the origin, properties which we should expect for degrees of freedom corresponding to the chiral fermions. In §4 we discuss these results in the context of chiral fermions appearing in singular M-theory compactifications. We suggest that our results support the absence of chiral fermions arising in the singular limit of M-theory compactified on a G_2 holonomy manifold asymptotic to a cone over $S^3 \times S^3$. Thus a conical singularity in itself is not enough to obtain a chiral four dimensional theory.

2 Dirac equation and all radial solutions

We will work with the simplest G_2 holonomy metric asymptotic to a cone over $S^3 \times S^3$ which has metric [23, 24]

$$ds^2 = \alpha(r)^2 dr^2 + \beta(r)^2 (\sigma^i - \frac{1}{2} \Sigma^i)^2 + \gamma(r)^2 \Sigma^i \Sigma^i, \quad (1)$$

$-\frac{1}{2}\epsilon^{ijk}\sigma^j \wedge \sigma^k$ and $d\Sigma^i = -\frac{1}{2}\epsilon^{ijk}\Sigma^j \wedge \Sigma^k$. The radial functions are

$$\alpha(r)^2 = (1 - \frac{r_0^3}{r^3})^{-1}; \quad \beta(r)^2 = \frac{1}{9}r^2(1 - \frac{r_0^3}{r^3}); \quad \gamma(r)^2 = \frac{1}{12}r^2. \quad (2)$$

The range of r is $r_0 \leq r < \infty$. It is useful to consider the vielbeins

$$\hat{e}^0 = \alpha dr; \quad \hat{e}^i = \beta(\sigma^i - \frac{1}{2}\Sigma^i); \quad \hat{e}^i = \gamma\Sigma^i. \quad (3)$$

We will use tangent space indices $i, j, k = 1, 2, 3$ and $\hat{i}, \hat{j}, \hat{k} = \hat{1}, \hat{2}, \hat{3} = 4, 5, 6$ and also $a, b, c, \dots = 0 \dots 6$. Spacetime indices will be μ, ν, \dots .

The massless Dirac equation corresponding to the metric (1) can be read off from the calculation of the connection one-form in [24],

$$\begin{aligned} \Gamma^a D_a \psi &= \Gamma^a \left(e_a - \frac{1}{4} \omega^{bc} \Gamma_{bc} \right) \psi \\ &= \left(\Gamma^a e_a + \frac{\beta'}{2\alpha\beta} \Gamma^{\hat{i}} \Gamma_{0\hat{i}} + \frac{\gamma'}{2\alpha\gamma} \Gamma^i \Gamma_{0i} + \epsilon^{ijk} \Gamma_{\hat{k}} \left(\frac{1}{8\beta} \Gamma_{\hat{i}\hat{j}} + \frac{\beta}{32\gamma^2} \Gamma_{ij} \right) \right. \\ &\quad \left. + \epsilon^{ijk} \Gamma_k \left(\frac{1}{8\gamma} \Gamma_{\hat{i}\hat{j}} - \frac{\beta}{16\gamma^2} \Gamma_{ij} + \frac{1}{8\gamma} \Gamma_{ij} \right) \right) \psi = 0, \end{aligned} \quad (4)$$

where as usual $\Gamma_{ab} = \frac{1}{2} [\Gamma_a, \Gamma_b]$ and e_a are vector fields dual to the vielbeins (3), $\langle e^b, e_a \rangle = \delta_a^b$. We use a convention for the gamma matrices such that the Clifford algebra is $\{\Gamma_a, \Gamma_b\} = -2\delta_{ab}$.

To solve this equation we need to choose a representation for the Euclidean Clifford algebra in seven dimensions. For this algebra there is a representation in which all the matrices are real. It is given by [28, 29]

$$(\Gamma_a)_{st} = c_{ast} + \delta_{as}\delta_{t7} - \delta_{at}\delta_{s7}, \quad (5)$$

where the spinor indices $s, t = 0 \dots 7$ and c_{ast} are zero if $s = 7$ or $t = 7$ and otherwise are the totally antisymmetric octonion structure constants, with

$$c_{016} = c_{052} = c_{043} = c_{142} = c_{135} = c_{236} = c_{456} = 1. \quad (6)$$

In this representation, the Majorana or reality condition is particularly simple

$$\psi^* = \psi. \quad (7)$$

The Dirac operator (4) takes Majorana (real) spinors to Majorana spinors. Therefore by expressing an arbitrary spinor in terms of two Majorana spinors

$$\psi = \psi^{(1)} + i\psi^{(2)}, \quad (8)$$

we may consider the Dirac equation for $\psi^{(1)}$ and $\psi^{(2)}$ separately.

solutions. However, the other half will be of exactly the same form but with a pure imaginary rather than real coefficient. Writing out ψ in components

$$\psi = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8), \quad (9)$$

where $f_1 \dots f_8$ are real functions. The Dirac equation thus becomes eight coupled equations for eight real functions. These equations are only tractable if one makes the radial ansatz

$$e_i \psi = e_i \psi = 0, \quad (10)$$

for $i = 1, 2, 3$. That is, we restrict the spinor to depend on r only.

Substituting (9) into the Dirac equation (4) one obtains the eight equations

$$\begin{aligned} r \frac{\partial f_8}{\partial r} + \frac{f_3}{2} - \sqrt{3} \sqrt{\frac{r^3}{r^3 - r_0^3}} f_6 - \frac{1}{4} \left(\frac{10r^3 - r_0^3}{r^3 - r_0^3} \right) f_1 + \frac{1}{4} \left(\frac{12r^3 - 3r_0^3}{r^3 - r_0^3} + 4\sqrt{3} \sqrt{\frac{r^3}{r^3 - r_0^3}} \right) f_8 &= 0, \\ r \frac{\partial f_3}{\partial r} + \frac{f_8}{2} - \sqrt{3} \sqrt{\frac{r^3}{r^3 - r_0^3}} f_1 + \frac{1}{4} \left(\frac{10r^3 - r_0^3}{r^3 - r_0^3} \right) f_6 + \frac{1}{4} \left(\frac{12r^3 - 3r_0^3}{r^3 - r_0^3} - 4\sqrt{3} \sqrt{\frac{r^3}{r^3 - r_0^3}} \right) f_3 &= 0, \\ r \frac{\partial f_6}{\partial r} - \frac{f_1}{2} - \sqrt{3} \sqrt{\frac{r^3}{r^3 - r_0^3}} f_8 + \frac{1}{4} \left(\frac{10r^3 - r_0^3}{r^3 - r_0^3} \right) f_3 + \frac{1}{4} \left(\frac{12r^3 - 3r_0^3}{r^3 - r_0^3} + 4\sqrt{3} \sqrt{\frac{r^3}{r^3 - r_0^3}} \right) f_6 &= 0, \\ r \frac{\partial f_1}{\partial r} - \frac{f_6}{2} - \sqrt{3} \sqrt{\frac{r^3}{r^3 - r_0^3}} f_3 - \frac{1}{4} \left(\frac{10r^3 - r_0^3}{r^3 - r_0^3} \right) f_8 + \frac{1}{4} \left(\frac{12r^3 - 3r_0^3}{r^3 - r_0^3} - 4\sqrt{3} \sqrt{\frac{r^3}{r^3 - r_0^3}} \right) f_1 &= 0, \\ r \frac{\partial f_7}{\partial r} - \frac{1}{4} \left(\frac{8r^3 + r_0^3}{r^3 - r_0^3} \right) f_2 + \frac{1}{4} \left(\frac{12r^3 - 3r_0^3}{r^3 - r_0^3} + 8\sqrt{3} \sqrt{\frac{r^3}{r^3 - r_0^3}} \right) f_7 &= 0, \\ r \frac{\partial f_2}{\partial r} - \frac{1}{4} \left(\frac{8r^3 + r_0^3}{r^3 - r_0^3} \right) f_7 + \frac{1}{4} \left(\frac{12r^3 - 3r_0^3}{r^3 - r_0^3} - 8\sqrt{3} \sqrt{\frac{r^3}{r^3 - r_0^3}} \right) f_2 &= 0, \\ r \frac{\partial f_5}{\partial r} + \frac{1}{4} \left(\frac{8r^3 + r_0^3}{r^3 - r_0^3} \right) f_4 + \frac{1}{4} \left(\frac{12r^3 - 3r_0^3}{r^3 - r_0^3} + 8\sqrt{3} \sqrt{\frac{r^3}{r^3 - r_0^3}} \right) f_5 &= 0, \\ r \frac{\partial f_4}{\partial r} + \frac{1}{4} \left(\frac{8r^3 + r_0^3}{r^3 - r_0^3} \right) f_5 + \frac{1}{4} \left(\frac{12r^3 - 3r_0^3}{r^3 - r_0^3} - 8\sqrt{3} \sqrt{\frac{r^3}{r^3 - r_0^3}} \right) f_4 &= 0. \end{aligned} \quad (11)$$

Note that these equations form three groups; for $\{f_2, f_7\}$, for $\{f_4, f_5\}$ and for $\{f_1, f_3, f_6, f_8\}$. To discuss the solutions, we need to distinguish the cases $r_0 \neq 0$ and $r_0 = 0$. In each case we obtain eight linearly independent solutions which, when combined with the other eight solutions of the same form but pure imaginary coefficient, are all the radial solutions to the Dirac equation (4).

Define $\bar{r} = r - r_0$. We are interested in whether the solutions are square integrable, i.e. L^2 , near the origin. For this we need the determinant of the metric (1) which near the origin is $\sqrt{g} \sim \bar{r}$ as $\bar{r} \rightarrow 0$. The behaviour of the eight solutions is summarised in the following table.

Independent solns.	As $\bar{r} \rightarrow 0$	As $\bar{r} \rightarrow \infty$	L^2 at origin?	Localised?
$\{f_2, f_7\}; \{f_4, f_5\}; \{f_1, f_3, f_6, f_8\}$	$\sim \bar{r}^0$	$\sim \bar{r}$	Yes	No
$\{f_2, f_7\}; \{f_4, f_5\}; \{f_1, f_3, f_6, f_8\}$	$\sim \bar{r}^{-3/2}$	$\sim \bar{r}^{-7}$	No	Yes
$\{f_1, f_3, f_6, f_8\}$	$\sim \bar{r}^0$	$\sim \bar{r}^0$	Yes	No
$\{f_1, f_3, f_6, f_8\}$	$\sim \bar{r}^{-3/2}$	$\sim \bar{r}^{-6}$	No	Yes

Where the notation in the table means that, for instance, there is a solution in which f_4 and f_5 go as a constant near the origin and as \bar{r} at infinity and all other radial functions are zero. The second and third columns show only the asymptotic power-law behaviour, not the numerical coefficients, which in general are not the same for the nonzero radial functions of each solution. The general radial solution is a linear superposition of these eight solutions with complex coefficients. None of the solutions are both L^2 at the origin and localised at the origin. We are not interested in the behaviour at infinity, although it is easy to see that none of the solutions are L^2 everywhere, in agreement with Lichnerowicz's theorem for noncompact manifolds.

2.2 Solutions with $r_0 = 0$

The form of the determinant at the origin is now $\sqrt{g} \sim r^6$ as $r \rightarrow 0$. The following table summarises the behaviour of the eight solutions.

Independent solns.	As $r \rightarrow 0$	As $r \rightarrow \infty$	L^2 at origin?	Localised?
$\{f_2, f_7\}; \{f_4, f_5\}; \{f_1, f_3, f_6, f_8\}$	$\sim r$	$\sim r$	Yes	No
$\{f_2, f_7\}; \{f_4, f_5\}; \{f_1, f_3, f_6, f_8\}$	$\sim r^{-7}$	$\sim r^{-7}$	No	Yes
$\{f_1, f_3, f_6, f_8\}$	$\sim r^0$	$\sim r^0$	Yes	No
$\{f_1, f_3, f_6, f_8\}$	$\sim r^{-6}$	$\sim r^{-6}$	No	Yes

The power-law asymptotics are in fact exact solutions in this case. We see that the asymptotic behaviour at infinity did not depend on r_0 , which is obvious from (2) anyway. Whilst the behaviour at the origin does change, it is qualitatively the same and still none of the solutions are both L^2 at the origin and localised at the origin. The possibility of the Lichnerowicz theorem not holding existed in this case because the manifold is singular.

Some non-radial solutions may be obtained by using relationships between eigenfunctions of different differential operators. This is possible because of the existence of covariantly constant spinors on special holonomy manifolds [9]. The idea is developed in [30] for four dimensional hyperKähler metrics and in [24] for massive eigenfunctions on G_2 and $\text{Spin}(7)$ metrics. The massless case we need is slightly more subtle. We will find solutions for the massless Dirac equation from solutions to the massless scalar equation. Suppose that ϕ satisfies

$$\Delta\phi = -\nabla_a\nabla^a\phi = 0, \quad (12)$$

and that η is the covariantly constant spinor, $D\eta = 0$. Then

$$\begin{aligned} \text{if } (\nabla_a\phi)\Gamma^a\eta = 0 \quad \text{then } \psi = \phi\eta \quad \text{solves } \Gamma^a\nabla_a\psi = 0, \\ \text{if } (\nabla_a\phi)\Gamma^a\eta \neq 0 \quad \text{then } \psi = (\nabla_a\phi)\Gamma^a\eta \quad \text{solves } \Gamma^a\nabla_a\psi = 0. \end{aligned} \quad (13)$$

One can also obtain solutions of the Dirac equation from solutions of the massless vector equation [24], but this is no easier to solve than the Dirac equation itself and so is not useful to us here.

The first thing we need to calculate is the covariantly constant spinor. This can be done [24, 31] by imposing the projection conditions

$$(\Gamma_{04} - \Gamma_{23})\eta = (\Gamma_{05} - \Gamma_{31})\eta = (\Gamma_{06} - \Gamma_{12})\eta = 0, \quad (14)$$

and then requiring η to have constant norm $\eta^\dagger\eta = 1$ and to be Majorana as in (7). This first condition is consistent because $D(\eta^\dagger\eta) = 0$ and the reality condition is necessary because the supersymmetry generator must be real for degrees of freedom to match up in the supergravity theory. All this determines η up to a sign, which can be chosen. One obtains

$$\eta^T = \frac{1}{2}(1, 0, -1, 0, 0, 1, 0, 1). \quad (15)$$

So the covariantly constant spinor has constant components. This will be generically true for cohomogeneity one metrics. The immediate consequence is that all r dependence of ψ will come from the scalar solution ϕ . By substitution it is easy to check that $D\eta = 0$ as required (cf. the constant solutions of §2.1 and §2.2 above).

The massless Klein-Gordon equation on the manifold can be written

$$\Delta\phi = -\frac{\delta^{ab}}{\sqrt{g}}\partial_\mu(e_a^\mu e_b^\nu\sqrt{g}\partial_\nu\phi) = 0, \quad (16)$$

where e_a^μ are the inverse vielbeins. This expression for the Hodge-de Rham operator is well suited for exploiting the symmetries of cohomogeneity-one metrics (1). It has been used to study the

[33]. For the metric (1), the equation becomes

$$\frac{1}{r^6(1 - r_0^3/r^3)} \partial_r(r^6(1 - r_0^3/r^3)^2 \partial_r \phi) + \frac{12}{r^2} J_i J_i \phi + \frac{3}{r^2} \frac{4 - r_0^3/r^3}{1 - r_0^3/r^3} J'_i J'_i \phi + \frac{12}{r^2} J_i J'_i \phi = 0, \quad (17)$$

where J_i are the vector fields dual to Σ^i and the J'_i are dual to σ^i . It follows that these vectors satisfy the algebra $[J_i, J_j] = \epsilon_{ijk} J_k$ and $[J'_i, J'_j] = \epsilon_{ijk} J'_k$. This means that as differential operators, $J_i J_i$, $J'_i J'_i$ and $J_i J'_i$ are simultaneously diagonalisable. Further, their eigenvalues are well known from elementary angular momentum theory. Acting on the simultaneous eigenfunctions Φ_{st}^{mn} we have

$$J_i J_i \Phi_{st}^{mn} = -m(m+1) \Phi_{st}^{mn}; \quad J'_i J'_i \Phi_{st}^{mn} = -n(n+1) \Phi_{st}^{mn}; \quad J_i J'_i \Phi_{st}^{mn} = -st \Phi_{st}^{mn}, \quad (18)$$

where $m, n \in \frac{1}{2}\mathbb{Z}^+ \cup \{0\}$ and $\{s, t\} \in \{m, n\} + \mathbb{Z}$ with $-m \leq s \leq m$ and $-n \leq t \leq n$. Note that the half integers are allowed because the group in question is $SU(2)$ and not $SO(3)$. We are not interested in the precise functional form of the angular eigenfunctions. Details and references can be found in [33]. One now decomposes the function into modes of the form $\phi = \Phi(r) \Phi_{st}^{mn}$. The equation (17) becomes

$$\frac{d^2}{dr^2} \Phi(r) + \frac{6r^2}{r^3 - r_0^3} \frac{d}{dr} \Phi(r) - 12r \left(\frac{m(m+1) + st}{r^3 - r_0^3} + n(n+1) \frac{4r^3 - r_0^3}{4(r^3 - r_0^3)^2} \right) \Phi(r) = 0. \quad (19)$$

This equation has four regular singular points, at $r^3 = r_0^3$ and at infinity. It does in fact have a solution in terms of hypergeometric functions, but these are only valid for $r < r_0$ whilst the range of r is $r \geq r_0$.

This reduction to a single ordinary differential equation was only possible because of the relatively large symmetry of the metric (1). More general G_2 metrics with less symmetry would have resulted in a Klein-Gordon equation in which the differential operators were not all simultaneously diagonalisable.

The behaviour at infinity is independent of r_0 and we have that as $r \rightarrow \infty$

$$\Phi(r) \sim r^{\frac{-1}{2}(5 \pm \sqrt{25 + 48Q})}, \quad (20)$$

where $Q = m(m+1) + n(n+1) + st \geq 0$. In the radial case, $Q = 0$, the two possible behaviours are just the $|\psi| \sim r^0$ and $|\psi| \sim r^{-6}$ solutions which we found in the previous section. For the second solution we use $|\psi| \sim \nabla_r \phi \sim \partial_r \phi$ as discussed above. The case of $Q \neq 0$ will be discussed below. The behaviour at the origin depends on whether r_0 is nonzero.

The behaviour as $\bar{r} \rightarrow 0$ for the non radial modes depends on whether the integer n is nonzero.

If $n \neq 0$ then as $\bar{r} \rightarrow 0$

$$\Phi(\bar{r}) \sim \bar{r}^{\frac{-1}{2}} \left(1 \pm \sqrt{1+4n(n+1)} \right). \quad (21)$$

The power of \bar{r} is nonzero and therefore $|\psi| \sim \bar{r}^{1/2} \partial_{\bar{r}} \phi \sim \bar{r}^{\frac{-1}{2}} \left(2 \pm \sqrt{1+4n(n+1)} \right)$. All of these solutions either are not localised or are not square integrable at the origin. Recall that the metric goes as $\sqrt{g} \sim \bar{r}$. The effect of having angular dependence on these solutions is to increase the degree of divergence at the origin of the localised solutions and increase the degree to which the non-localised solutions increase away from the origin.

And if $n = 0$ then as $\bar{r} \rightarrow 0$

$$\Phi(\bar{r}) \sim k_1 \bar{r}^0 + k_2 \bar{r}^{-1}. \quad (22)$$

These correspond to spinors behaving like $|\psi| \sim \bar{r}^0$ and $|\psi| \sim \bar{r}^{-3/2}$. The first does not decay and the second is not L^2 at the origin. This set of solutions does not change its radial behaviour as the angular dependence changes. The behaviour found is again consistent with the results of the previous section.

3.2 Solutions with $r_0 = 0$

The solution as $r \rightarrow 0$ is the same as in the $r \rightarrow \infty$ limit of (20). We now consider the effect of angular dependence. All of the solutions are nonconstant in r and so

$$|\psi| \sim \partial_r \phi \sim r^{\frac{-1}{2}} (7 \pm \sqrt{25+48Q}). \quad (23)$$

Again, all of these solutions either are not localised or are not square integrable at the origin, because $Q \geq 3/4$. The metric now goes as $\sqrt{g} \sim r^6$. Again, the effect of including angular dependence is to make the solutions divergent at the origin more divergent and the solutions that grow away from the origin grow faster.

4 Discussion

We have found radial solutions that are constant, solutions decaying away from the origin and solutions growing away from the origin. None of these satisfied the two properties of being both localised and square integrable at the origin. The first of these is expected for degrees of freedom emerging as a homology cycle collapses and the second is necessary for the solution to

solutions we found.

The results admit three possible interpretations. Firstly, that the degrees of freedom corresponding to chiral fermions localised near the singularity correspond to nonradial solutions of the Dirac equation that we have not found. Secondly, that solving the Dirac equation is not the way to exhibit these degrees of freedom in any case. Thirdly, that the G_2 manifold asymptotic to a cone over $S^3 \times S^3$ does not give chiral fermions in the singular limit, unlike the cones over $SU(3)/U(1) \times U(1)$ and \mathbb{CP}^3 .

The third of these possibilities is the most interesting and perhaps the most plausible. It links nicely with the results from dualities and anomaly cancellation discussed in the introduction. It also fits with the picture of extra massless modes coming from collapsed membranes and the fact that the G_2 metric asymptotic to a cone over $S^3 \times S^3$ has zero second homology group. Solving the Dirac equation on other G_2 metrics that are known to give chiral fermions appears to be more challenging, but could lend further support to this possibility.

Many thanks to Gary Gibbons for useful suggestions. Thanks also to Ruben Portugues, James Sparks and Tibra Ali for discussions. The author is funded by the Sims scholarship.

References

- [1] E. Witten, *Search for a Realistic Kaluza-Klein Theory*, Nucl. Phys. **B 186** (1981) 412.
- [2] P. Hořava and E. Witten, *Heterotic and Type I String Dynamics from Eleven Dimensions*, Nucl. Phys. **B460** (1996) 506, [hep-th/9510209](#).
- [3] P. Hořava and E. Witten, *Eleven-Dimensional Supergravity on a Manifold with a Boundary*, Nucl. Phys. **B475** (1996) 94, [hep-th/9603142](#).
- [4] M. Atiyah and E. Witten, *M-theory Dynamics on a Manifold of G_2 Holonomy*, [hep-th/0107177](#).
- [5] M. Cvetič, G. Shiu and A.M. Uranga, *Three-Family Supersymmetric Standard-like Models from Intersecting Brane Worlds*, Phys. Rev. Lett. 87 (2001) 201801, [hep-th/0107143](#).
- [6] M. Cvetič, G. Shiu and A.M. Uranga, *Chiral Four-Dimensional $N = 1$ Supersymmetric Type IIA Orientifolds from Intersecting D6-Branes*, Nucl. Phys. **B615** (2001) 3, [hep-th/0107166](#).

- [8] E. Witten, *Anomaly Cancellation On G_2 -Manifolds*, hep-th/0108165.
- [9] D. Joyce, *Compact Manifolds With Special Holonomy*, Oxford University Press (2000).
- [10] S.S. Gubser, *TASI lectures: special holonomy in string theory and M-theory*, hep-th/0201114.
- [11] M.J. Duff, B.E.W. Nilsson and C.N. Pope, *Kaluza-Klein Supergravity*, Phys. Rept. **130** (1986) 1.
- [12] M.J. Duff, *M-theory on Manifolds of G_2 Holonomy: the First Twenty Years*, hep-th/0201062.
- [13] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Supersymmetric M3-branes and G_2 Manifolds*, Nucl. Phys. **B620** (2002) 3, hep-th/0106026.
- [14] M. Cvetič, G.W. Gibbons, J.T. Liu, H. Lü and C.N. Pope, *A New Fractional D2-brane, G_2 Holonomy and T-duality*, hep-th/0106162.
- [15] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Cohomogeneity One Manifolds of $Spin(7)$ and G_2 Holonomy*, hep-th/0108245.
- [16] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Orientifolds and slumps in G_2 and $Spin(7)$ metrics*, hep-th/0111096.
- [17] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *M-theory Conifolds*, hep-th/0112098.
- [18] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *A G_2 Unification of the Deformed and Resolved Conifolds*, hep-th/0112138.
- [19] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, *Gauge Theory at Large N and New G_2 Holonomy Metrics*, Nucl. Phys. **B611** (2001) 179, hep-th/0106034.
- [20] A. Brandhuber, *G_2 Holonomy Spaces from Invariant Three Forms*, hep-th/0112113.
- [21] A. Bilal, J-P. Derendinger and K. Sfetsos, *(Weak) G_2 holonomy from self-duality, flux and supersymmetry*, hep-th/0111274.
- [22] R. Cleyton and A. Swann, *Cohomogeneity-one G_2 -structures*, math.DG/0111056.

- holonomy*, Duke Math. J. **58** (1989) 829.
- [24] G.W. Gibbons, D.N. Page and C.N. Pope, *Einstein metrics on S^3 , \mathbb{R}^3 and \mathbb{R}^4 bundles*, Commun. Math. Phys. **127** (1990) 529.
 - [25] E. Witten, *String Theory Dynamics in Various Dimensions*, Nucl. Phys. **B443** (1995) 85, [hep-th/9503124](#).
 - [26] M. Bershadsky, C. Vafa and V. Sadov, *D-Strings on D-Manifolds*, Nucl. Phys. **B463** (1996) 398, [hep-th/9510225](#).
 - [27] M. Atiyah, J. Maldacena and C. Vafa, *An M-theory Flop as a Large N Duality*, J.Math.Phys. **42** (2001) 3209, [hep-th/0011256](#).
 - [28] E. Corrigan, C. Devchand, D.B. Fairlie and J. Nuyts, *First-Order Equations for Gauge Fields in Spaces of Dimension Greater than Four*, Nucl. Phys. **B214** (1983) 452.
 - [29] M. Günaydin and F. Gürset, *Quark structure and octonions*, J. Math. Phys. **14** (1973) 1651.
 - [30] S.W. Hawking and C.N. Pope, *Symmetry breaking by instantons in supergravity*, Nucl. Phys. **B146** (1978) 381.
 - [31] M. Cvetič, H. Lü and C.N. Pope, *Brane Resolution Through Transgression*, Nucl. Phys. **B600** (2001) 103, [hep-th/0011023](#).
 - [32] G.W. Gibbons and N.S. Manton, *Classical and Quantum Dynamics of BPS Monopoles*, Nucl. Phys. **B274** (1986) 183.
 - [33] S. Mignemi, *Classical and quantum motion in an Eguchi-Hanson space*, J.Math.Phys. **32** (1991) 3047.